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ASYMPTOTIC STABILITY AND ESTIMATE OF THE REGION OF ATTRACTION IN CERTAIN SYSTEMS WITH AFTEREFFECT†

V. S. SERGEYEV

Moscow

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The problem of stability in the first approximation is solved for systems with aftereffect by integrodifferential equations of the Volterra type, using ideas and methods developed by Lyapunov [1] in the theory of the stability of differential equations and later developed further in the context of the theory of non-linear oscillations [2, 3]. © 1997 Elsevier Science Ltd. All rights reserved.

Problems of the stability and existence of solutions of various types of equations with aftereffect and functional-differential equations have been considered in a great many publications, the results of which are reflected fairly well in [4–8]. The method of Lyapunov functionals [4, 5, 9] is widely employed in stability analysis.

In [10–13] we considered integrodifferential equations of the Volterra type, which are exponentially stable in the first approximation, with non-linearities represented by holomorphic functions of the variables of the problem and of given functionals. The general solution in the neighbourhood of an asymptotically stable zero solution was represented by the series of Lyapunov's First Method. A majorizing equation was constructed by using a majorant for the non-linearity in the form of a power series with non-negative coefficients (Cauchy majorant). The majorizing equation was used to obtain estimates for the region of attraction.

In this paper an attempt will be made to improve the estimates for the attraction region by using Lyapunov majorants [3] as majorizing functions and constructing general solutions by the method of successive approximations. At the same time the class of non-linear functions occurring in the equation will be enlarged by dropping the holomorphicity condition.

Suppose the system with aftereffect is given by the equation

$$\frac{dx}{dt} = A(t)x + \int_{t_0}^{t} K(t,s)x(s)\,ds + F(x,y,t), \quad x \in R^n, \quad y \in R^m \tag{1}$$

in which the continuous $n \times n$ matrices $A(t)$ and $K(t)$ are defined on sets $I = \{t \in R: t \geq t_0\}$ and $I_1 = \{(t, s) \in R^2: t_0 \leq s \leq t < +\infty\}$, respectively, and have bounded elements. The variable y is defined by the relation

$$y = \int_{t_0}^{t} K_1(t,s) f(x,s)\,ds \tag{2}$$

where the kernel $K_1(t, s)$ is of the same type as $K(t, s)$ and the vector function $f(x, t) = \mathrm{col}(f_1, \ldots, f_m)$, defined on a set $B_1(x) \times I$, where $B_1(x) = \{x \in R^n: \|x\| < R_1\}$ for some $R_1 > 0$, is a C^1 function of x, continuous and bounded with respect to $t \in I$ and such that $f(0), t) \equiv 0$. The vector function $F(x, y, t) = \mathrm{col}(F_1, \ldots, F_n)$: $B_2(x, y) \times I \to R^n$, where $B_2(x, y) = \{x \in R^n, y \in R^m: \|x\| < R_1, \|y\| < R_2\}$ for given $R_i > 0$ $(i = 1, 2)$, is a C^1 function of x, y and a continuous and bounded function of $t \in I$ such that $F(0, 0, t) \equiv 0$. In addition, $F(x, y, t)$ is such that if the variable x is replaced by εx (this being done also in the expression for y; ε is a parameter), then it becomes a vector function F' with the property $\partial F'/\partial\varepsilon|_{\varepsilon=0} = 0$.

Let us consider the Cauchy problem with initial value $x(t_0) = x_0$. Henceforth we will use the following notation. Let B be a matrix (vector). Then the symbol B_* will denote the matrix (vector) each of whose elements (components) is the absolute value of the corresponding element (component) of B.

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We introduce vector functions similar to the Lyapunov majorants of [3]. These will be majorants $\Phi(x, y) = \mathrm{col}(\Phi_1, \ldots, \Phi_n) > 0$, $\varphi(x) = \mathrm{col}(\varphi_1, \ldots, \varphi_m) > 0$ for $F(x, y, t)$ and $f(x, t)$, respectively, i.e. vector functions with positive components which are monotone increasing functions of each of the coordinates x_k, y_j of x, y and whose first derivatives are positive, continuous, bounded, monotone increasing functions in $B_2(x, y)$ (or in $B_1(x)$), and moreover

$$
\begin{gathered}
F_*(x,y,t) \leqslant \Phi(u, v), \quad f_*(x,t) \leqslant \varphi(u) \\
\left(\frac{\partial F_k(x,y,t)}{\partial x}\right)_* \leqslant \frac{\partial \Phi_k(u,v)}{\partial u}, \quad \left(\frac{\partial F_k(x,y,t)}{\partial y}\right)_* \leqslant \frac{\partial \Phi_k(u,v)}{\partial v} \\
\left(\frac{\partial f_j(x,t)}{\partial x}\right)_* \leqslant \frac{\partial \varphi_j(u)}{\partial u}, \quad k = 1,2,\ldots,n; \quad j = 1,2,\ldots,m \\
\Phi(0,0) = 0, \quad \varphi(0) = 0, \quad \frac{\partial \Phi}{\partial u} = 0, \quad \frac{\partial \Phi}{\partial v} = 0 \quad \text{for} \quad u = 0, \ v = 0
\end{gathered}
\tag{3}
$$

for all $x_* \leqslant u, y \leqslant v, t \in I$ and $(u, v) \in B_2(u, v)$.

Let $X(t, t_0)$ be a fundamental matrix of the linearized equation (1) such that $X(t_0, t_0) = E_n$.

We assume that the following conditions hold for $t \in I$, $(t, s) \in I_1$

$$
X_*(t,t_0) \leqslant C\exp[-\alpha(t-t_0)], \quad K_{1*}(t,\tau) \leqslant C_1\exp[-\beta(t-\tau)] \tag{4}
$$

where $\alpha > 0$, $\beta > 0$ are constants and $C > 0$, $C_1 > 0$ are positive constant $n \times n$ and $m \times n$, matrices, respectively.

We introduce the notation

$$
\begin{gathered}
V(x,p) = pC_1\varphi(x), \quad \nu = \min(\alpha, \beta) \\
M(\gamma) = \frac{1}{\beta}\left(\frac{\gamma}{\beta}\right)^{\gamma/(\beta-\gamma)}, \quad m = \lim_{\gamma \to \nu} M(\gamma)
\end{gathered}
\tag{5}
$$

Theorem 1. Assume that Eq. (1), (2) satisfy the conditions listed above for the continuity and differentiability of the functions $A(t)$, $K(t, s)$, $K_1(t, s)$, $F(x, y, t)$, $f(x, t)$ and that conditions (4) are satisfied. Let $\Phi(x, y)$, $\varphi(x)$ be given Lyapunov majorants, so that inequalities (3) hold, in addition, let

$$
\begin{gathered}
\Phi_k(\varepsilon u, \varepsilon v) \leqslant \varepsilon^{1+\delta}\Phi_k(u,v), \quad k = 1,2,\ldots,n \\
\varphi_j(\varepsilon u) \leqslant \varepsilon\varphi_j(u), \quad j = 1,2,\ldots,m
\end{gathered}
\tag{6}
$$

for any ε $(0 \leqslant \varepsilon \leqslant 1)$, some $\delta = \text{const} > 0$ and any non-negative u and v such that $(u, v) \in B_2(u, v)$.

Then

1. the trivial solution of Eq. (1), (2) is exponentially stable;
2. the boundary Γ of a set $G \subset B_1(x_0)$ belonging to the region of attraction is given by the equations

$$
\begin{gathered}
\det(E_n - NC\,d\Phi'(u)/du) = 0, \quad u = C(x_{0*} + N\Phi'(u)) \\
(\Phi'(u) = \Phi(u, V(u,m)), \ N = 1/\alpha)
\end{gathered}
\tag{7}
$$

Proof. Fix a number $\gamma > 0$ such that $\gamma < \nu$, $\alpha - \gamma(1 + \delta) \neq 0$. Equation (1), (2) with initial value x_0 is equivalent to an integral equation [7]

$$
x(t) = X(t,t_0)\,x_0 + \int_{t_0}^{t} X(t,\tau)\,F(x(\tau), y(\tau), \tau)\,d\tau \tag{8}
$$

Let us represent the solution of Eq. (8), (2) in the form of successive approximations $x^{(k)}(t), y^{(k)}(t)$ $(k = 1, 2, \dots)$, where $x^{(1)}(t) = X(t, t_0)x_0$.

For any $k \geqslant 1$ and $t \in I$ we have the inequalities

$$x_*^{(k)}(t) \leqslant \exp[-\gamma(t-t_0)]\,u^{(k)}, \quad y_*^{(k)}(t) < \exp[-\gamma(t-t_0)]\,V(u^{(k)}, M_1) \tag{9}$$

where $u^{(k)} > 0$ are constant vectors.

The proof proceeds by induction. If $k = 1$, the assertion is true for $x_*^{(1)}(t)$ by virtue of (4). For $y_*^{(1)}(t)$, taking (3) and (4) into consideration, we obtain the estimate

$$y_*^{(1)}(t) \leqslant \int_{t_0}^{t} C_1 \exp[-\beta(t-\tau)]\exp[-\gamma(\tau-t_0)]\,\varphi(u^{(1)})\,d\tau < \exp[-\gamma(t-t_0)]\,V(u^{(1)}, M_1), \tag{10}$$

$$M_1 = (\beta-\gamma)^{-1}$$

Suppose that inequalities (9) hold for $k = 1, 2, \dots, s-1$. Then for $k = s$ $(s > 1)$ it follows from the equality

$$x^{(s)}(t) = X(t,t_0)\,x_0 + \int_{t_0}^{t} X(t,\tau)\,F(x^{(s-1)}(\tau), y^{(s-1)}(\tau))\,d\tau \tag{11}$$

and from (9), (3), (4) and (6) that

$$\begin{aligned} x_*^{(s)}(t) &\leqslant C\exp[-\gamma(t,t_0)]\,x_{0*} + C\int_{t_0}^{t} \exp[-\alpha(t-\tau)-(1+\delta)\gamma(\tau-t_0)]\,\Phi(u^{(s-1)}, V(u^{(s-1)}, M_1))\,d\tau \leqslant \\ &\leqslant \exp[-\gamma(t-t_0)]\,C(x_{0*} + N_1\Phi(u^{(s-1)}, V(u^{(s-1)}, M_1))) \end{aligned} \tag{12}$$

where the constant $N_1 > 0$ satisfies the following inequality for $t \in I$

$$N_1 \geqslant (\exp[-\gamma\delta(t-t_0)] - \exp[-(\alpha-\gamma)(t-t_0)])/[\alpha-(1+\delta)\gamma]$$

An estimate analogous to (10) can be established for $y^{(s)}(t)$, so that inequalities (9) will be true for $k = s$.

Let $\tilde{u}^{(k)} > 0$ be a monotone increasing sequence, independent of t, such that $\tilde{u}^{(k)} \geqslant x_*^{(k)}(t)$, implying that the sequence $x^{(k)}(t)$ is convergent. Define.

$$\tilde{u}^{(1)} = Cx_{0*}, \quad \tilde{u}^{(k)} = C\left(x_{0*} + \frac{1}{\alpha}\Phi(\tilde{u}^{(k-1)}, V(\tilde{u}^{(k-1)}, M))\right), \quad k = 2, 3, \dots \tag{13}$$

The sequence $\tilde{u}^{(k)}$ is majorizing for $x^{(k)}(t)$; it converges to the unique positive solution $u = u(x_0)$ of the equation

$$u = C\left(x_{0*} + \frac{1}{\alpha}\Phi(u, V(u, M))\right) \tag{14}$$

which always exists in view of the properties of the functions $\Phi(x, y)$, $\varphi(x)$ [3].

Indeed, applying an estimate analogous to (10), without isolating the exponentially decreasing term, we obtain $x_*^{(1)}(t) \leqslant V(\tilde{u}^{(1)}, M)$, and also

$$(x^{(2)}(t) - x^{(1)}(t))_* \leqslant C\int_{t_0}^{t} \exp[-\alpha(t-\tau)]\,F_*(x^{(1)}(\tau), y^{(1)}(\tau))\,d\tau < \tilde{u}^{(2)} - \tilde{u}^{(1)}$$

If it is true for all $k = 1, 2, \dots, s-1$ that

$$(x^{(k)}(t) - x^{(k-1)}(t))_* < \tilde{u}^{(k)} - \tilde{u}^{(k-1)}$$

then, using the properties (3) of the Lyapunov majorant ([3], Chap. II), equalities (11) and (13), and the estimate $y_{*}^{(k)}(t) \leqslant V(\bar{u}^{(k)}, M)$, we conclude that for $k = s$

$$(x^{(s)}(t)-x^{(s-1)}(t))_{*} \leqslant C\int_{t_0}^{t}\exp[-\alpha(t-\tau)](F(x^{(s-1)}(\tau),y^{(s-1)}(\tau))-$$

$$-F(x^{(s-2)}(\tau),y^{(s-2)}(\tau)))_{*}d\tau < \frac{C}{\alpha}(\Phi(\bar{u}^{(s-1)},V(\bar{u}^{(s-1)},M))-$$

$$-\Phi(\bar{u}^{(s-2)},V(\bar{u}^{(s-2)},M))) = \bar{u}^{(s)}-\bar{u}^{(s-1)}$$

The quantity $M = M(\gamma)$ of (5) is a monotone decreasing function of γ in the interval $(0, \nu)$, tending to a finite limit as $\gamma \to \nu$.

Putting the constant M in (14) equal to $m = \lim M(\gamma) \to \nu$, we obtain the second equation of (7). A solution $u = u(x_0)$ of this equation exists for all $x_0 \in G$, where the boundary Γ of G is defined, as follows from [2, 3], by Eqs (7). Thus, the solution $x(t)$ of Eq. (1), (2) with initial value $x_0 \in G$ may be obtained as the limit of uniformly convergent successive approximations for all $t \in I$, and by (9) $x(t) \to 0$ exponentially as $t \to +\infty$. This completes the proof of the theorem.

In specific cases, along with the second equation of (7), one can construct majorizing equations that enable one to establish wider estimates for the region of attraction. The following proposition is an example.

Theorem 2. Under all previous assumptions about the properties of the functions $A(t)$, $K(t, s)$, $K_1(t, s)$, $F(x, y, t)$, $f(x, t)$, suppose that inequalities (4) hold, subject to the condition $\alpha < \beta$, and that $\Phi(x, y)$, $\varphi(x)$ are Lyapunov majorants which are polynomials (or convergent power series) with non-negative coefficients, such that the terms of lowest degree in $\Phi(x, y)$ are of order $p \geqslant 2$.

Then the assertions of Theorem 1 with Eqs (7) are true, with the following substitutions

$$\Phi'(u) = \Phi(u, V(u,m_1)), \quad m_1 = (\beta-\alpha)^{-1}, \quad N = \frac{1}{\alpha}p^{p/(1-p)}$$

The functions $\Phi(u, v)$, $\varphi(u)$ of Theorem 2 need not satisfy any additional condition like (6), and the majorizing sequence $\bar{u}^{(k)} \geqslant x^{(k)}(t)$ may be a sequence defined by relations analogous to (13) with $1/\alpha$ replaced by N and M by M_1, where

$$N \geqslant N'(t) = (\exp[-p\gamma(t-t_0)] - \exp[-\alpha(t-t_0)])/(\alpha - p\gamma) \tag{15}$$

These relations are based on inequalities (12), without isolating the exponential factor.

The majorizing equation $u = \lim_{k\to\infty} \bar{u}^{(k)}$ becomes

$$u = C(x_{0*} + N\Phi(u, V(u, m_1)))$$

where N is the maximum of $t \in I$ (15) over $N'(t)$ for $\gamma = \alpha$.

Note that, of course, the estimate for the region of attraction ensuing from Theorem 2 is worth using only provided $\beta - \alpha$ is not small. In particular, it is effective when Eq. (1) is independent of y.

Let us investigate the equation

$$\frac{dx}{dt} = A(t)x + \int_{t_0}^{t} K(t,s)\,x(s)\,ds + F(x,y,z,t), \quad z \in R^l \tag{16}$$

in which x, y, $A(t)$, $K(t, s)$ have the same properties as in (1), (2), and which depends on an analytic functional represented by Volterra–Fréchet series of multiple integrals [14, 15, Chaps 1 and V]), so that, if $z = \mathrm{col}(z_1, \ldots, z_l)$, then

$$z_i(t) = \sum_{k=1}^{\infty}\sum_{j(k)=1}^{n}\int_{t_0}^{t}\ldots\int_{t_0}^{t} K_i^{j(k)}(t,s_1,\ldots,s_k)\,x_{j_1}(s_1)\ldots x_{j_k}(s_k)\,ds_1\ldots ds_k \quad i=1,2,\ldots,l \tag{17}$$

where $j(k)$ is a sequence of indices $j_1, \ldots, j_k$ and the kernels $K_i^{j(k)}(t, s_1, \ldots, s_k)$: $I_k \to R$ are continuous and bounded, with

$$I_k = \{(t, s_1, \ldots, s_k) \in R^{k+1}: t_0 \leqslant s_j \leqslant t < +\infty,\ j = 1, 2, \ldots, k\}$$

The function $F(x, y, z, t)$: $B_3(x, y, z) \times I \to R^n$ (the domain B_3 is analogous to B_2 for (1)) is a C^1 function with respect to x, y, z, bounded as a function of $t \in I$, $F(0, 0, 0, t) \equiv 0$, and if x is replaced by εx (this being done in (2) and (17), also) it becomes a function F' such that $\partial F'/\partial \varepsilon\,|_{\varepsilon=0} = 0$.

Let us assume that the integral kernels in (17) satisfy the following exponential estimate

$$|K_i^{j(k)}(t, s_1, \ldots, s_k)| \leqslant C_2 \exp\left[-\sum_{p=1}^{k} \varkappa_p (t - s_p)\right] \tag{18}$$

$$\varkappa_p > 0,\quad C_2 = \text{const} > 0$$

where $\varkappa_p \geqslant \varkappa$ ($p = 1, 2, \ldots$) for some $\kappa > 0$.

Let $\Phi(u, v, w) = \operatorname{col}(\Phi_1, \ldots, \Phi_n)$ be a Lyapunov majorant for $F(x, y, z, t)$ which satisfies conditions in u, v, w analogous to conditions (3) for $\Phi(u, v)$.

Let $W(x, p, q) \in R^l$ denote the vector with equal components

$$w_0(x, p, q) = pC_2 \sum_{k=1}^{n} x_k \left(1 - q \sum_{k=1}^{n} x_k\right)^{-1};\quad p, q - \text{const}$$

and let $M_2(\gamma)$ be the quantity $M(\gamma)$ of (5) with β replaced by κ; we also define $\nu = \min(\alpha, \beta, \kappa)$ and

$$M_3(\gamma) = \frac{1}{\beta}\left(\frac{\gamma(1+\rho)}{\beta}\right)^L,\quad L = \frac{\gamma(1+\rho)}{\beta - \gamma(1+\rho)},\quad m_i = \lim_{\gamma \to \nu} M_i(\gamma),\quad i = 2, 3$$

Theorem 3. Suppose Eq. (16), (2), (17) satisfies conditions (4) and (18), and let $\Phi(u, v, w)$, $\varphi(u)$ be given Lyapunov majorants for the functions $F(x, y, z, t)$, $f(x, t)$, satisfying the inequalities

$$\Phi_k(\varepsilon u, \varepsilon v, \varepsilon w) \leqslant \varepsilon^{1+\delta} \Phi_k(u, v, w)$$

$$\varphi_j(\varepsilon u) \leqslant \varepsilon^{1+\rho} \varphi_j(u),\quad j = 1, 2, \ldots, m;\quad k = 1, 2, \ldots, n$$

for any ε ($0 \leqslant \varepsilon \leqslant 1$), certain constants $\delta > 0$, $\rho \geqslant 0$ and any non-negative u, v and w such that $(u, v, w) \in B_3(u, v, w)$.

Then

1. the trivial solution of Eq. (16) is exponentially stable;
2. the boundary Γ of a region G belonging to the region of attraction is defined by Eqs (7) with $\Phi'(u) = \Phi(u, V(u, m_3), W(u, m_2, m_2))$.

The proof is analogous to that of Theorem 1.

Estimating the successive approximations $z_i^k(t)$ of the functionals $z_i(t)$ in (17) and using (18) and (9), we obtain, for example, for $k = 1$

$$|z_i^{(1)}| \leqslant \sum_{k=1}^{\infty} \sum_{j(k)=1}^{n} \int_{t_0}^{t} \ldots \int_{t_0}^{t} C_2 \exp\left[-\sum_{p=1}^{k} (\varkappa_p (t - s_p) - \gamma(s_p - t_0))\right] u_{j_1}^{(1)} \ldots u_{j_k}^{(1)} ds_1 \ldots ds_k <$$

$$< C_2 \sum_{k=1}^{\infty} \sum_{j(k)=1}^{n} M_2^{k-1}(\gamma)\, u_{j_1}^{(1)} \ldots u_{j_k}^{(1)} (\varkappa - \gamma)^{-1} \exp[-\gamma(t - t_0)] =$$

$$= w_0(u^{(1)}, (\varkappa - \gamma)^{-1}, M_2(\gamma)) \exp[-\gamma(t - t_0)] \tag{19}$$

where $M_2(\gamma)$ is the maximum over $t \in I$ of the function $(\exp[-\gamma(t - t_0)] - \exp[-\varkappa(t - t_0)])/(\varkappa - \gamma)$. Inequalities similar to (19) hold for $z_i^k(t)$ for all $k \geqslant 1$. The construction of a majorizing equation uses estimates for $z_i^k(t)$ of the same type as (19), but without isolating the exponential term; to be precise: $|z_i^{(k)}| < w_0(u^{(k)}, M_2(\gamma), M_2(\gamma))$.

Note that $M_2(\gamma)$ is a monotone decreasing function of γ in the interval $(0, \kappa)$ taking values between κ^{-1} and $(e\kappa)^{-1}$. The function $\gamma > 0$, whose singularities are removable, also decreases monotonically as $M_3(\gamma)$.

In applications one often limits attention to segments of series (17). In that case the majorants for $z_i(t)$ obtained in this way may be polynomials.

Let us consider some simple examples of constructing estimates for regions of attraction.

Example 1. Consider the equation

$$\frac{dx}{dt}=\int_{t_0}^{t} K(t-s)\,x(s)\,ds-\sin x+by^2,\quad y=\int_{t_0}^{t} K_1(t-s)\,x(s)\,ds \tag{20}$$

$$x, y \in R^1,\quad b=\text{const}$$

where $K(t)$ and $K_1(t)$ satisfy the assumptions of Theorem 1. Suppose that all the roots λ_s of the characteristic equation

$$\lambda+1-\int_0^{\infty} K(s)\exp(-\lambda s)\,ds=0$$

corresponding to (20) are such that $\operatorname{Re}\lambda_s < \lambda' < 0$. Then the numbers α and C occurring in the first inequality of (4) exist.

By Theorem 1, if $u = u_0$, $\mu = \mu_0$ is a positive solution of the system of equations

$$u=C(\mu+N\Phi'(u)),\quad CN\,d\Phi'(u)/du=1 \tag{21}$$

then the inequality $|x_0| < \mu_0$ provides an estimate for the region of attraction. As Lyapunov majorant we take $\Phi(x,y) = x^3/6 + |b|y^2$. Then for (21) we have $\Phi'(u) = u^3/6 + |b|(mC_1u)^2$, $m = (e\beta)^{-1}$ if $\beta \leqslant \alpha$, and $m = \beta^{-1}(\alpha\beta^{-1})^{m4}$, $m_4 = \alpha(\beta) - \alpha)^{-1}$ if $\alpha < \beta$. Solving system (21), we obtain $\mu_0 = u_0/C - \Phi(u_0)/\alpha$, where $u_0 = -c + (c^2 + 2\alpha/C)^{1/2}$, $c = 2|b|m^2C_1^2$.

A remark is in order regarding this example. Let us use the above Lyapunov majorant $\Phi(x, y)$ to investigate an equation with a non-linearity that is not a holomorphic function of x, given in some neighbourhood of zero such that $|x| < R_1$ (R_1 = const > 0). Suppose that the right-hand side of Eq. (20) is retained for $|x| \leqslant \varepsilon_0$ ($0 < \varepsilon_0 < R_1$), but for $\varepsilon_0 < |x| < R_1$ the non-linear terms x on the right that depend on $z = \psi_1(x)$ are replaced by quadratic terms $\psi_1(x)$ such that the parabola $z = \psi_1(x)$ passes through points $x = 0, z = 0$ and $x = \varepsilon_0, z = \varepsilon_0 - \sin\varepsilon_0$, preserving the continuity at $x = \varepsilon_0$ of the first derivative for the non-linear function thus constructed. Then the estimate obtained in this example for the region of attraction remains valid. It is clear, however, that if one determines an estimate of the region of attraction for Eq. (20), thus transformed, in the region $|x| < \varepsilon_0$, where the right-hand side of the equation is holomorphic, then for some small ε_0 a narrower estimate of the region of attraction will be obtained. This remark stresses how important it is to drop the holomorphicity requirement in the general case.

Example 2. Consider a rigid body in a uniform gravitational field of force, capable of revolving about a fixed horizontal axis OO_1 which is the axis of torsion of two visco-elastic rods attached to the body (as a support). It is assumed that the axis OO_1 is not deformable. The second end of each rod is fixed. Let ϑ denote the angle in the plane orthogonal OO_1 between the downward vertical and a straight line passing through OO_1 and the mass centre, which lies at a distance r from the axis. In equilibrium $\vartheta = 0$. When the body rotates, the rods are twisted. The moment about OO_1 of the forces applied to the body by the rods may be expressed as [15]

$$M_\upsilon=-k\vartheta+\int_{t_0}^{t} K'(t-s)\,\vartheta(s)\,ds,\quad k=\text{const}>0$$

Let J be the axial moment of inertia of the body and mg its weight. The equation of perturbed motion of the body about the equilibrium position

$$J\frac{d^2\vartheta}{dt^2}=-k\vartheta+\int_{t_0}^{t} K'(t-s)\,\vartheta(s)\,ds-mgr\sin\vartheta \tag{22}$$

may be written as a second-order system, by setting $\vartheta' = d\vartheta/dt$, and we can represent it in the integral form (8). One equation of this system is

$$\vartheta(t)=x_{11}(t-t_0)\,\vartheta_0+x_{12}(t-t_0)\,\vartheta_0'+\int_{t_0}^{t} x_{12}(t-s)\,F(\vartheta(s))\,ds \tag{23}$$

$$\vartheta_0=\vartheta(t_0),\quad \vartheta_0'=\vartheta'(t_0),\quad F(\vartheta)=m_0(\vartheta-\sin\vartheta),\quad m_0=mgr/J$$

where $(x_{ij}(t - t_0)) = X(t - t_0)$ is a fundamental matrix. We will assume that the first inequality of (4) holds with $C = (c_{ij})$ $(i, j = 1, 2)$ and apply Theorem 2 directly to Eq. (23). Equations (7), with $p = 3$, $N = 3^{-3/2}/\alpha$ and $\Phi(u_1) = m_0u_1^3/6$, where $u_1 \gg \vartheta$, are

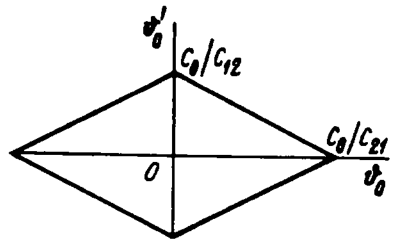

Fig. 1.

$$u_1 = c_{11}|\vartheta_0| + c_{12}|\vartheta_0'| + \frac{d}{3}u_1^3, \quad 1 - du_1^2 = 0, \quad d = \frac{c_{12}m_0}{6\sqrt{3}\alpha}$$

The required estimate for Eq. (22) is

$$c_{11}|\vartheta_0| + c_{12}|\vartheta_0'| < c_0, \quad c_0 = \frac{2}{3}d^{-1/2}$$

The form of the region (23) is shown in the Fig. 1.

Incidentally, the monotonicity condition imposed on the Lyapunov majorants and their first derivatives with respect to all variables may be slightly weakened, as follows from the general properties of majorizing equations [3, Chap. 1].

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